# THE EQUILIBRIUM BETWEEN MATTER AND RADIATION

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Abstract

The equilibrium concentration of electrons and protons is recalculated on the basis of Dirac's new theory of the nature of the proton; it is found to be exceedingly small, of the same order of magnitude as had been found in previous calculations.

THERE have recently been a number of attempts to calculate the equilibrium between matter and radiation in the universe.[1,2,3] There are two difficulties in these calculations; one of these is the generalization of energy and entropy which must be made before the equilibrium state of the entire universe can be treated; this problem has been discussed in particular by Tolman. The other difficulty is the assignment of entropy to a perfect crystal at the absolute zero; it cannot be regarded as entirely certain that this entropy is equal to that of a perfect vacuum, as is assumed. Nevertheless, all calculations agree in yielding for the final equilibrium state one in which the ratio of the energy in the form of matter to that in the form of radiation is exceedingly small; this ratio is dominated by the exponential factor $e^{-mc^2/kT}$ which completely wipes out the effect of all the other factors; the final state of the universe indicated by these calculations is thus one in which there is practically no matter left.

Quite recently Dirac[4] has proposed a theory of the nature of the proton which seems to call for a new calculation of the equilibrium concentration of matter. Briefly stated, Dirac's theory is that the fundamental unit of matter is the electron; in addition to the usual states in which the electron is observed, there are others in which its mass is negative; in these states its total energy is negative, and in fact becomes more negative as its velocity increases. These negative energy electrons are attracted by ordinary electrons, but ordinary electrons are nevertheless repelled by them. All these properties correspond to solutions of the wave equation of the electron which were previously known, but generally considered extraneous. Dirac proposes the hypothesis that space contains great numbers of these negative energy electrons, which obey the Fermi statistics; the states of lowest energy (highest velocity) are therefore full, but a few of the states of low velocity are not

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[1] Stern, Zeits. f. Elektrochem. **31,** 448 (1925); Trans. Far. Soc. **21,** 477 (1925–6).

[2] Tolman, Proc. Nat. Acad. Sci. **12,** 670 (1926); **14,** 353 (1928).

[3] Zwicky, Proc. Nat. Acad. Sci. **14,** 592 (1928).

[4] Dirac, Proc. Roy. Soc. **126A,** 360 (1930).

occupied; these gaps constitute irregularities in the normal arrangement of space, and are observable. It is evident that the gaps will attract ordinary electrons, and be attracted by them; they will also have in effect a positive energy, since they correspond to the absence of a particle of negative energy.

The most natural assumption is that the total number of electrons in the universe is just equal to the number of cells of negative energy; then at the absolute zero every electron would be in the lowest possible state, there would be complete uniformity throughout space, and the universe would be observationally empty. The problem to be solved is simply that of finding the distribution of electrons among the various cells at higher temperatures; we shall confine our solution to a finite region in which there is flat space-time. The problem then differs from the usual applications of the Fermi statistics chiefly in that there are two continuous ranges of energy values. We divide these ranges into intervals in the usual way, and write for the number of cells in the $s^{th}$ interval of the positive energy range

$$Q_s = 4\pi V/h^3(2m)^{3/2}E_s^{1/2}\Delta E \tag{1}$$

and for the number in the $t^{th}$ interval of the negative energy range

$$Q_t = 4\pi V/h^3(2M)^{3/2}E_t^{1/2}\Delta E. \tag{2}$$

We have included in these formulae a factor 2 arising from the spin, which we suppose exists for all energies, and in (2) we use the observed proton mass $M$; there is some doubt about the correct procedure at this point; $-M$ is what the chemist would call the partial molal (or partial molecular) mass of the electrons of true mass $-m$; it is the right value to use for the change in mass produced in a system by creating the first proton in it; when another proton is created sufficiently close a different value will be needed. The result of our calculations will be that the concentration of matter is exceedingly small, and $M$ therefore is certainly the correct mass for most purposes, though possibly not for ours; it would not make any important change in the results if $m$ did replace $M$ in (2). Also it is unnecessary to use the correct relativistic expressions for (1) and (2), since all cells of large positive kinetic energy are empty, and all those of large negative kinetic energy are full; the number of cells of these kinds will not be correctly given by our result, but this number does not concern us.

The number of distribution of $N_s$ particles among $Q_s$ cells, using the Fermi statistics, is known to be $Q_s!/N_s!(Q_s-N_s)!$ and hence the number of distributions for the entire system, the numbers $N_s$ and $N_t$ being specified, is

$$W = \prod_s \frac{Q_s!}{N_s!(Q_s-N_s)!} \prod_t \frac{Q_t!}{N_t!(Q_t-N_t)!}\,. \tag{4}$$

This is to be a maximum subject to the conditons of conservation of charge and of energy. Proceeding in the usual way we have

$$\sum_s \{-\log N_s + \log (Q_s - N_s)\}\delta N_s + \sum_t \{-\log N_t + \log (Q_t - N_t)\}\delta N_t = 0 \quad (4)$$

$$\sum_s \delta N_s + \sum_t \delta N_t = 0 \quad (5)$$

$$\sum_s (E_s + mc^2)\delta N_s - \sum_t (E_t + Mc^2)\delta N_t = 0. \quad (6)$$

Using multipliers $e^\alpha$ and $e^\beta$ for (5) and (6) we add these three equations and then require each term in the two sums to vanish. Upon rearranging the necessary conditions we have

$$N_s = \frac{Q_s}{e^{\alpha + (E_s + mc^2)\beta} + 1} \quad (7)$$

$$N_t = \frac{Q_t}{e^{\alpha - (E_t + Mc^2)\beta} + 1} \cdot \quad (8)$$

It is easily shown in the usual manner that $\partial E/\partial S = 1/k\beta$ and since thermodynamics requires $\partial E/\partial S = T$,we have as always in this type of calculation

$$\beta = 1/kT. \quad (9)$$

For positive values of $\alpha$ the first term in the denominator of (7) will be very large and (7) may be written approximately

$$N_s = Q_s e^{-\alpha - (E_s + mc^2)/kT}. \quad (10)$$

For values of $\alpha$ not too large the first term in the denominator of (8) will be extremely small and we will have

$$N_t = Q_t \{1 - e^{\alpha - (E_t + Mc^2)/kT}\}. \quad (11)$$

We now determine $\alpha$ from the state of electrification of the system. The condition for neutrality is

$$\sum_s N_s = \sum_t (Q_t - N_t) \quad (12)$$

which becomes

$$\sum_s Q_s e^{-\alpha - (E_s + mc^2)/kT} = \sum_t Q_t e^{\alpha - (E_t + Mc^2)/kT}. \quad (13)$$

Upon introducing the values of $Q_s$ and $Q_t$ and replacing the sum by an integral we find

$$(2m)^{3/2} e^{-\alpha - mc^2/kT} \int_0^\infty e^{-E/kT} E^{1/2} dE = (2M)^{3/2} e^{\alpha - Mc^2/kT} \int_0^\infty e^{-E/kT} E^{1/2} dE \quad (14)$$

or

$$e^\alpha = (m/M)^{3/4} e^{(M-m)c^2/2kT}. \quad (15)$$

This value of $\alpha$ is evidently such that the approximations (10) and (11) are justified. Upon inserting (15) and (1) into (10) and integrating we obtain for the number of electrons of positive energy

$$\begin{aligned} N_+ &= \frac{4\pi V}{h^3}(4mM)^{3/4}e^{-(m+M)c^2/2kT}\int_0^\infty e^{-E/kT}E^{1/2}dE \\ &= 2V\left(\frac{2\pi(mM)^{1/2}kT}{h^2}\right)^{3/2}e^{-(m+M)c^2/2kT}. \end{aligned} \tag{16}$$

This then is the number of electrons and also the number of protons which the system contains. It is very closely similar to Stern's result for the number of particles of mass $m$

$$N = V\left(\frac{2\pi mkT}{h^2}\right)^{3/2}e^{-mc^2/kT}. \tag{17}$$

The vanishingly small amount of matter permitted by this equation has already been pointed out, and our result may be regarded as in some measure supporting the view that if any matter is to be preserved in the final equilibrium of the universe it must be rescued by the tendency of matter toward aggregation. But the evidence of astronomy suggests that the stars are constantly gaining matter in the form of dust and meteors, transforming it into radiation and sending it back into space; this may mean, of course, that the universe cannot save its matter by any device and that it is steadily fading away. On the other hand the evidence of the cosmic rays may be supposed to indicate that in the depths of space radiation is converted back into matter; if this process is occurring it can only mean that the foregoing calculation, and all others of a similar nature, are utterly incorrect.